FROM HYPERELLIPTIC TO SUPERELLIPTIC CURVES

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The theory of elliptic and hyperelliptic curves has been of crucial importance in the development of algebraic geometry. Almost all fundamental ideas were first obtained and generalized from computations and construction carried out for elliptic or hyperelliptic curves. Examples are elliptic or hyperelliptic integrals, theta functions, Thomae's formula, the concept of Jacobians, etc. Some of the classical literature on the subject as well as the seminal work of Jacobi focus almost entirely on hyperelliptic curves. These lectures follow mostly the line from [2].

So what is so special about a hyperelliptic curve? To begin with, a generic curve in the hyperelliptic locus admits a cyclic Galois cover to the projective line. This cover, which is called the *hyperelliptic projection* is of degree $n = 2$ and its branch points determine the curve in question (up to isomorphism). Hence, studying hyperelliptic curves over algebraically closed fields amounts to studying degree two covers of the projective line. In other words, it is more convenient to think of a hyperelliptic curve $\mathcal{X}$ as a degree two branched covering $f : \mathcal{X} \to \mathbb{P}^1$.

A natural generalization of the above is to study degree $n \geq 2$ cyclic Galois covers. This means that for a curve $\mathcal{X}$ with automorphism group Aut $(\mathcal{X})$ there is a cyclic group $H = \langle \tau \rangle$ normal in Aut $(\mathcal{X})$ such that the quotient $\mathcal{X}/H$ is isomorphic to $\mathbb{P}^1$. Such curves $\mathcal{X}$ are called *superelliptic curves*. The automorphism τ is called the *superelliptic automorphism* of $\mathcal{X}$.

There are two different ways of studying an algebraic curves; function fields $L/k(x)$ or coverings of $\mathbb{P}^1$. The goal of this book is the study of algebraic curves via the coverings of $\mathbb{P}^1$ in the tradition of Riemann, Clebsch, Hurwitz, Severi, Grothendieck, Fulton, Fried, et al. We focus on the cyclic covers of $\mathbb{P}^1$ and by doing that we investigate the natural generalization of the theory of hyperelliptic curves to superelliptic curves. Our goal is to highlight the theories that can be extended and all the open problems that come with this generalization. We investigate the correspondence from the group theory data of the cover $f : \mathcal{X} \to \mathbb{P}^1$, via Riemann Existence Theorem (RET), to the field of moduli of $\mathcal{X}$, relations among the thetanulls on Jac $(\mathcal{X})$ and the branch points of $f : \mathcal{X} \to \mathbb{P}^1$, the Hurwitz spaces of coverings with ramification structure as that of f. This enables us to use the full machinery of group theory to study covers and get more information about the arithmetic aspects of curves.

As we will see during these lectures, many aspects of the theory of the hyperelliptic curves can be extended to all superelliptic curves. For example, such curves have affine equations (over an algebraically closed field) of the form $y^n = f(x)$, the list of full automorphism groups of such curves can be fully determined, as can be determined their equations and in most cases such curves can be defined over their field of moduli. Most importantly the full machinery of classical invariant theory of binary forms can be used to determine their isomorphism classes. It is such theory that makes the study of the moduli space of such curves much more concrete than for general curves. More importantly the invariant theory connects the theory of superelliptic curves to the weighted projective moduli spaces. It is the weighted moduli space and the weighted heights introduced in [1] which make the study of the arithmetic of the moduli space possible.

1. Lecture 1: Preliminaries on curves and their automorphisms

In Lecture 1, we will briefly describe some basic generalities of covering spaces and monodromy, algebraic curves and their function fields, Weierstrass points and their weights, and full automorphism groups of curves. Such material prepares us to naturally introduce the superelliptic curves as a generalization of hyperelliptic curves. Such curves have a degree $n \geq 2$, cyclic Galois covering $\pi : \mathcal{X}_g \to \mathbb{P}^1$. We denote the branched points of this cover by the roots of some polynomial $f(x)$ and show that the curve has equation $y^n = f(x)$. We also determine the list of possible full automorphism group of a superelliptic curve $\mathcal{X}_g$ of genus $g \geq 2$. Furthermore, we study the Weierstrass points of superelliptic curves and show that they are projected to the roots of $f(x)$ as in the hyperelliptic case.

2. Lecture 2: Families of covers and moduli spaces

In Lecture 2, we focus on families of covers of $\mathbb{P}^1$ with fixed ramification structure. Using group theory to study such families leads us to Hurwitz spaces, braid group action, and a full stratification of the moduli space of curves. We further study the loci of superelliptic curves in the moduli space. We briefly introduce the moduli space of curves $\mathcal{M}_{g_0,r}$ and its Deligne-Mumford compactification $\overline{\mathcal{M}}_{g_0,r}$. Then we focus on points of the $\overline{\mathcal{M}}_{g_0,r}$ which correspond to curves with automorphisms. We discuss the inclusions between such loci and give the complete stratification of the moduli space for genii $g = 3, 4$.

Since isomorphism classes of superelliptic curves $y^n = f(x)$ defined over a field k correspond to $\mathrm{GL}_2(k)$-equivalence classes of degree $d = \deg f$ binary forms, we focus on the classical invariant theory of binary forms. We give the preliminaries of classical invariant theory of degree d binary forms including the ring of invariants $\mathcal{R}_d$ and explicitly determine the ring of invariants for binary sextics and binary octavics. The ring of invariants $\mathcal{R}_d$ is a graded ring. Its projective variety is a weighted projective variety. Further we study the weighted moduli space $\mathbb{WP}^n_{\mathfrak{w}}(k)$ for a given set of weights $\mathfrak{w} = (q_0, \dots, q_n)$. Weighted projective spaces are a much more convenient way to study superelliptic curves. Weighted greatest common divisor and weighted height are introduced in order to study the arithmetic properties of $\mathbb{WP}^n_{\mathfrak{w}}(k)$. The weighted heights enable us to "sort" and "count" points in the weighted projective space. For such heights it is true the Northcott's theorem and all the height machinery a-la Weil. This opens a new and exiting direction of research on the arithmetic of weighted varieties.

3. Lecture 3: Superelliptic Jacobians

In Lecture 3, we study the Jacobians of superelliptic curves. Theta functions of superelliptic curves are discussed. The classical Thomae's formula for hyperelliptic curves relates the branch points of the hyperelliptic projection to the thetanulls. It is a natural question to ask if such formula can be generalized to cyclic or superelliptic curves. During the last decade ha been some considerable progress on this direction. We describe a Thomae like formula for cyclic curves. Furthermore, we study Jacobian varieties and briefly describe Mumford's representation of divisors and Cantor's algorithm for addition of points on a hyperelliptic Jacobian. Generalizations of this approach to superelliptic Jacobians are discussed in the rest of the chapter. We generalize the cord and secant method of addition on elliptic and hyperelliptic curves to all superelliptic curves. Superelliptic curves have nice bases of the space of holomorphic differentials, which give a monomial basis for the function field of the curve over the ground field. When this basis is ordered according to the order of this monomials at a fixed point $P \in \mathcal{X}$, then it provides a way to define a curve $\mathcal{Y}$ similar to a line adding two points on an elliptic curve.

4. LECTURE 4: INTEGRAL MINIMAL MODELS

In Lecture 4 we will be focused on the arithmetic of superelliptic curves. Given a point in the moduli space $\mathcal{M}_g$ or $\mathbb{WP}(g)$, defined over a number field K, the first arithmetic question is whether the corresponding curve is defined over K. As we know, this is not true in general. However, for superelliptic curves we are able to determine in the majority of cases if there is a curve defined over K. Once a curve is defined over a number field K, then it is defined over its ring of integers $\mathcal{O}_K$. The next natural question is if such equation can be found explicitly and whether it can be made minimal. We study minimal models of superelliptic curves when a moduli point is given. This is well known, due to work of Tate, for elliptic curves and Liu for genus two. We describe briefly Tate's algorithm. For superelliptic curves we we say that a curve has minimal model when it has a minimal moduli point . We give necessary and sufficient condition on the set of invariants of the curve that the curve has a minimal model. Moreover an algorithm is provided how to find such minimal model. Further we introduce Neron-Tate heights and how they are applied to superelliptic Jacobians. Neron-Tate models of superelliptic Jacobians will be discussed.

These lectures are intended for advanced graduate students or young mathematicians who want to work in the arithmetic of superelliptic curves. We assume that the reader has familiarity with basic theoretical aspects of algebraic curves, Riemann surfaces, Jacobian varieties, which are part of the math folklore in the subject.

REFERENCES

[1] L. Beshaj, J. Gutierrez, and T. Shaska, *Weighted greatest common divisors and weighted heights*, Journal of Number Theory (2019).

[2] A. Malmendier and T. Shaska, *From hyperelliptic to superelliptic curves*, Albanian J. Math. **13** (2019), no. 1, 107–200. MR3978315